Problem Set 4 for Biomath 202: Due May 31, 2017

1. a. Starting from the definition of the Gamma function \( \Gamma(x) = \int_0^\infty d\lambda \lambda^{x-1} e^{-\lambda} \), use integration by parts to show that \( \Gamma(x+1) = x\Gamma(x) \).

b. With \( x \) still an integer, derive an expression for \( \Gamma(\frac{x}{2} + 1) \) by deriving \( \Gamma(1/2) = \sqrt{\pi} \). You can use known results for Gaussian integrals.

c. Use the recursion relation in 1a to prove that \( \Gamma(\varepsilon) \sim 1/\varepsilon \) for \( \varepsilon<<1 \), and thus that in the limit \( \varepsilon->0 \), \( \Gamma(0) = \infty \).

2. As an individual grows, its growth in mass, \( M \), can be described by

\[
\frac{dM}{dt} = \mu M
\]

and metabolic rate, \( B \), by

\[
\frac{dB}{dt} = \beta B
\]

where \( \beta \) and \( \mu \) are constants. Show that this leads to a power law relationship between metabolic rate and body mass. Identify what the exponent is. Discuss what this process means for cellular metabolic rate, \( B_c \), and cellular mass, \( m_c \), for the case that the exponent is not equal to 1.

3. Consider the Barabasi model of preferential attachment, but now add two new vertices per time step. Each new vertex still gets attached to \( m \) other vertices at the time step it is first introduced. Does this change in the rules affect the scaling exponent for the connectivity, \( k_i \), of vertex, \( i \), with time \( t \) and if so, how?

4. Repeat problem 3, but now let the number of new vertices introduced at time \( t \) scale like a power law, as \( V(t) = V_0 t^p \). How does this affect the scaling of the connectivity, \( k_i \), of vertex, \( i \), with time \( t \)? Is it still a power law? If not, what is form does it take? What is its behavior at long times as \( t \) approaches infinity?
(Hint: Be careful when computing the total number of edges at time $t$.)

5. a. Derive the general result for a geometric series $\sum_{k=s}^{n} x^k$. What is the analogous integral and what would it evaluate to? How do the integral and summation results compare with one another? Can you make them agree in some limit?

b. What is the sum of $\sum_{k=s}^{n} \frac{1}{k(k+1)(k+2)(k+3)\ldots(k+j)}$ for arbitrary $j$?

What is the analogous integral and what would it evaluate to? How do the integral and summation results compare?

c. Can you derive a simple expression for $\sum_{k=s}^{n} \frac{1}{k}$? What would the analogous integral be and can you evaluate it?

6. In this problem, we will derive an approximation for the gamma function (and thus also for the factorial) for large values of $x$.

a. Argue that $\ln(\lambda x e^{-\lambda})$ is sharply peaked near $\lambda = x$.

b. Let $\lambda = x + \varepsilon$, where $\varepsilon \ll x$ and expand the function around the peak to obtain the approximation $\lambda x e^{-\lambda} \sim x^x e^{-x} e^{-\frac{1}{2}x^2}$.

c. Use this expression as the integrand in the formula for $\Gamma(x + 1)$, and make one final approximation to express the remaining integral as a Gaussian integral. Your final result should be the approximation $\Gamma(x + 1) \sim x^x e^{-x} \sqrt{2\pi x}$. This approximation method is known as steepest descents. This result is known as Stirling’s formula.