Modeling Vascular Networks with Applications

Instructor: Van Savage
Spring 2015 Quarter
1. Warm up for elastic vessel walls by solving equations when both the axial and radial components of the velocity are non-zero. This allows bulges and contractions in the radial movement of flow down the tube.

2. Understand equations for vessel walls

3. Look at equations for vessel walls as thin-wall approximation

4. Use boundary conditions to link vessel walls and fluid flow

5. Calculate volume flow rate, impedance, and Korteweg-Moens velocity in z-direction for case of massless walls
Movement of fluid in axial and radial directions
Movement of fluid in axial and radial directions

Velocity vector has two components

\[ \vec{v} = (v_r, 0, v_z) \]

We still assume rotational symmetry, so no \( v_\theta \) and no dependence of other velocity components on \( \theta \). Still avoiding real problem of turbulence by neglecting non-linearities, which we will continue to do. Our equations become

\[
\rho \frac{dv_z}{dt} + \frac{dp}{dz} = \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{\partial^2 v_z}{\partial z^2} \right]
\]

\[
\rho \frac{dv_r}{dt} + \frac{dp}{dr} = \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_r}{\partial r} \right) + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} \right]
\]
Two more equations

Equation of continuity

\[ \nabla \cdot \vec{v} = 0 \Rightarrow \frac{1}{r} \frac{\partial}{\partial r} (rv_r) = -\frac{\partial v_z}{\partial z} \]

Also, by taking divergence of our Navier-Stokes equations, we find

\[ \nabla^2 p = 0 \Rightarrow \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + \frac{\partial^2 p}{\partial z^2} = 0 \]

This is Laplace’s equation for the pressure, and it is one of the most well known PDEs that exists. Instead of separating off the pressure term as before, we can now solve Laplace’s equation to use a more general method.
Using separation of variables and Fourier transforms as before, we obtain

\[
p(r, z, t) = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk e^{i(\omega t - kz)} \tilde{p}(k, \omega) J_0(ikr)
\]
Now add on an inhomogenous piece to cancel the pressure gradient for general solution

\[ v_z(r,z,t) = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk e^{i(\omega t - kz)} \left[ \hat{v}_z^H(k,\omega) J_0(ik' r) + \hat{v}_z^I(k,\omega) J_0(ikr) \right] \]

\[ \frac{d}{dz} \text{of this just brings down an } ik \text{ that can be absorbed into the arbitrary function and this is } k \text{ not } k' \text{ because pressure just depends on } k. \]
Using same tricks to find $v_r$

$$v_r(r,z,t) = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk e^{i(\omega t - kz)} \left[ \hat{v_r}^H(k,\omega)J_1(ik^1 r) + \hat{v_r}^I(k,\omega)J_1(ikr) \right]$$

$d/dr$ of this brings down an $ik$ that can be absorbed into the arbitrary function and changes the Bessel function to zeroth order. Again, this is $k$ not $k'$ because pressure just depends on $k$. 
Substituting our solutions back into the original equations we find

\[ \hat{v}_z^I(k,\omega) = \frac{k}{\omega\rho} \tilde{p}(k,\omega) = \hat{v}_r^I(k,\omega) \]
Using the continuity equation we find

\[ \hat{v}_r^H (k, \omega) = \frac{k}{k'} \hat{v}_z^H (k, \omega) \]
After reducing the number of unknowns we have

\[ p(r,z,t) = \int d\omega \int dk e^{i(\omega t - kz)} \tilde{p}(k,\omega) J_0(ikr) \]

\[ v_z(r,z,t) = \int d\omega \int dk e^{i(\omega t - kz)} [\hat{v}_z^H(k,\omega) J_0(ik' r) + \frac{k}{\omega \rho} \tilde{p}(k,\omega) J_0(ikr)] \]

\[ v_r(r,z,t) = \int d\omega \int dk e^{i(\omega t - kz)} \left[ \frac{k}{k'} \hat{v}_z^H(k,\omega) J_1(ik' r) + \frac{k}{\omega \rho} \tilde{p}(k,\omega) J_1(ikr) \right] \]
Equations for elastic vessels walls
Infinitesimal volume element

\[ R \, \theta \, d\xi \]

Lagrangian coordinates for walls
Infinitesimal volume element

\[ \delta V = hR \delta \theta \delta z \]

\[ \delta m = \rho_w \delta V \]
Forces due to contact with boundaries such as vessel walls and among different regions of fluid flow:

More generally, for x and y mixed forces:

\[ p_{xy} = S_{xy} = \mu \left( \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \]

Across all planes of force, these are our stressed and strains:

\[ p_{ij} = S_{ij} = \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \]
Diagram of all shear stresses and strains in analog to those for Navier-Stokes
Forces along vessel walls and element of contact between wall and fluid

Force within vessel wall due to elasticity

\[ F = \delta S_{zz} h R \delta \theta = \frac{\delta S_{zz}}{\delta z} \delta z h R \delta \theta = \frac{\delta S_{zz}}{\delta z} h(R \delta \theta \delta z) \]

Shear force caused by friction between moving fluid and tube wall

\[ F = \tau_w R \delta \theta \delta z \]
Forces in radial direction balanced by forces from tissue and elasticity of wall and pressure from fluid at vessel-fluid boundary.

Radial stress pushing vessel wall towards center:

\[ F = -S_{rr}R\delta\theta\delta z \]

Fluid pressure within vessel, which is net difference between pressure inside and outside of vessel walls:

\[ F = p_w R\delta\theta\delta z \]
Forces in rotational direction are symmetric and cancel
Newton’s law

We are now in Lagrangian coordinates, and directly following mass elements because coordinates are tied to actual wall and not some background reference frame.

Net force against wall in the axial direction is

\[ F = \rho_w R \delta \theta \delta z h \frac{\partial^2 \xi_z}{dt^2} \]

Net force against wall in the radial direction is

\[ F = \rho_w \ h R \delta \theta \delta z \frac{\partial^2 \xi_r}{dt^2} \]
Equations of motion

Putting all of this together for the axial direction, we have

\[ \rho_w h \frac{\partial^2 \xi_z}{dt^2} = h \frac{\delta S_{zz}}{\delta z} + \tau_w \]

Putting all of this together for the radial direction, we have

\[ \rho_w h \frac{\partial^2 \xi_r}{dt^2} = p_w - S_{rr} \]
Equations of motion

Often for these problems, the results below are quoted as empirical facts at the boundary of the vessel wall, \( r=R \).

\[
\rho_w h \frac{\partial^2 \xi_z}{dt^2} = Eh \left( \frac{\partial^2 \xi_z}{dz^2} + \frac{\sigma}{R} \frac{\partial \xi_r}{dz} \right) - \mu \left( \frac{\partial v_r}{dz} + \frac{\partial v_z}{dr} \right)
\]

Poisson’s ratio deals with non-homogeneity of vessel wall material

\[
\rho_w h \frac{\partial^2 \xi_r}{dt^2} = p_w - Eh \left( \frac{\sigma}{R} \frac{\partial \xi_z}{dz} + \frac{\xi_r}{R^2} \right)
\]

Young’s modulus is a measure of stiffness
(Read Feynman’s lectures for more about elasticity)
Equations for elastic vessel walls by analogy with fluid equations

\[ \rho \frac{\partial \mathbf{v}}{\partial t} = \mu \nabla^2 \mathbf{v} - \nabla p \]  

\[ \rho_w \frac{\partial^2 \xi}{\partial t^2} = E \nabla^2 \xi - \nabla p_w \]

Note the equations for the flow are in terms of velocity and for the wall are in terms of the position coordinates. This has to do with choice of coordinates for solid versus fluid. Vessel walls are incompressible, so divergence of position vector is zero as for velocity of fluid. Note that we have written it ignoring Poisson’s ratio by analogy with fluid.
More generally, equations of motion can be written as the Navier equation for vessel walls

\[ \rho_w h \frac{\partial^2 \xi_i}{dt^2} = \partial_j \sigma_{ij} \]

with the definitions

Stress and strain tensor as for viscosity term in Navier-Stokes

\[ \sigma_{ij} = 2Ee_{ij} + (\lambda e_{kk} - P)\delta_{ij} = 2Ee_{ij} + -P\delta_{ij} \]

Pressure as from Navier-Stokes

\[ e_{ij} = \frac{1}{2}(\partial_i \xi_j + \partial_j \xi_i) \]