Computational and Systems Biology Course 186—Modeling of Biological Systems by Connecting Biological Knowledge and Intuition with Mathematics and Computing

Instructor: Van Savage
Winter 2017 Quarter
Monday and Wednesday, 2-3:50pm
3/6/2017
Biological Power Laws, Scaling, and Allometry
Metabolic rate--“Fire of life”--power for maintenance, growth, and reproduction

heat loss:
metabolic rate=heat loss rate$\propto$ surface area

Stefan-Boltzmann law

isometry--shape stays same

allometry--shape changes

metabolic rate=heat loss rate$\propto$ surface area$\propto (volume)^{2/3}$

Animals are not isometric, so not a good null hypothesis.
Mass dependence of metabolic rate

Basal Rate, slope of 3/4, unexpected

Hemmingsen, 1966
Mammalian Basal Metabolic Rate

\[ y = 0.737x - 1.739 \]

\[ r^2 = 0.99 \]

Whole Plant Xylem Flux

\[ y = 0.736x - 2.230 \]

\[ r^2 = 0.91 \]

Annual Biomass Production Per Individual


Population Density

Ubiquitous 1/4-power Scaling
Like Kepler’s Laws

Scaling Exponent

Rates at the cellular, individual, and population level for many different taxa scale in this way. Many times and lengths also scale.

Compiled from Peters (1983); Savage et al., Func. Eco. 2004
Review of power laws
1. Types of Power Laws

\[ y = ax^b \]

1. Physical laws--planetary orbits, parabolic motion of thrown objects, classical forces, etc. (these are really idealized notions and do not exist in real world)

2. Scaling relations--relate two fundamental parameters in a system like lifespan to body mass in biology (physical laws are special/strong case)

3. Statistical distributions
Identifying Power Laws

\[ \ln y = b \ln x + \ln a \]

Linear plot: slope=b and intercept=ln A

- Need big range on x- and y-axes to determine power laws because this minimizes effects of noise and errors
- Can give good measure of \( b \), the exponent
- \( r^2 \) is property of data and measures how much variance in \( y \) is explained by variance in \( x \). It is NOT really a measure of goodness of fit!
Examples of Power Laws
Parabolic Motion (Type 1)
Rates related to vascular networks (Type 2)

Word usage (Type 3)
Word Usage (Type 3)

![Log-log graph showing word frequency against log rank, with a table of unigrams and their meanings.]

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Web Sites (Type 2)
Identifying Power Laws

• Maximum likelihood methods are good for identifying power laws if used in correct way

• Whether to curve fit in linear or logarithmic space depends on distribution of errors because regressions make assumptions about these: homoscedascity->variance in \( y \) is independent of value of \( x \)

• Parabolic motion

• Body size (for population, variance in body size or heart rate varies linearly with \( x \))--logarithmic space
Self Similarity and Fractals
Imagine taking a picture of a smaller piece and magnifying it, and then it looks like the original part.

- Once a useful process is found in nature, it tends to be used over and over again—physics constrains possible processes and evolution tends to maintain it because most mutations are harmful.
Other examples of self similarity
Hunting the Hidden Dimension special on Nova:

Benoit Mandelbrot
How do you think this one was generated?
One movie just for fun

http://www.ericbigas.com/fractals/tilm/
Tropic_Isle_Level_Mandel.mpg
Power Laws ⇔ Self Similarity

Equation form for self similarity:

\[ f(\lambda x) = \lambda^p f(x) \]

\[ f(x) = ax^p \]

\[ f(\lambda x) = a(\lambda x)^p = \lambda^p ax^p = \lambda^p f(x) \]

\[ p\lambda^{p-1} f(x) = \frac{df(\lambda x)}{d\lambda} = \frac{d(\lambda x)}{d\lambda} \frac{df(\lambda x)}{d(\lambda x)} = x \frac{df(\lambda x)}{d(\lambda x)} \]

Free to choose \( \lambda = 1 \)

\[ x \frac{df(x)}{dx} = pf(x) \Rightarrow \frac{df}{f} = p \frac{dx}{x} \]

\[ \Rightarrow f(x) = ax^p \]
Fixed Points and Universality
Dynamical systems flow relative to fixed points

What is functional form as fixed point is approached? This describes region and dynamics that are relevant for many scientific questions.
Non-power-law functions often behave as power laws near critical points

- Other functions commonly occur in nature: \(e^x\), \(\sin(x)\), \(\cosh(x)\), \(J_\nu(x)\), \(\text{Ai}(x)\)
- These functions can generally be expressed in Taylor or power series near critical points (phase transitions, etc.). When \(x\) is close to \(x^*\), difference is small, and first term dominates.

\[
f(x) = \sum_{k=p}^{N} \frac{(x - x^*)^k}{k!} \left[ \frac{d^k f}{dx^k} (x - x^*) \right] \sim C(x - x^*)^p
\]

- \(p\)-exponent of leading-order term
- Any functions with the same first term in their series expansion behave the same near critical points, which is of great physical interest, even if they behave very differently elsewhere. Source of universality classes.
Near $x=0$, both of these functions scale like $x$!
Dimensional Analysis and Power Laws
Dimensional Analysis

- Often used in physics
- For reasons given thus far, many processes should scale as a power law.
- Given some quantity, \( f \), that we want to determine, we need to intuit what other variables on which it must depend, \( \{x_1, x_2, \ldots, x_n\} \).
- Assume \( f \) depends on each of these variables as a power law.
- Use consistency of units to obtain set of equations that uniquely determine exponents.

\[
f(x_1, x_2, \ldots, x_n) = x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}
\]
Example 1: Pythagorean Theorem

• Hypotenuse, c, and smallest angle, \( \theta \), uniquely determine right triangles.
• Area = \( f(c, \theta) \), DA implies Area = \( c^2 g(\theta) \).

\[
\begin{align*}
\text{Area of whole triangle} &= \text{sum of area of smaller triangles} \\
&= a^2 g(\theta) + b^2 g(\theta) = c^2 g(\theta) \\
\Rightarrow a^2 + b^2 &= c^2
\end{align*}
\]
Example 2: Nuclear Blast

• US government wanted to keep energy yield of nuclear blasts a secret.

• Pictures of nuclear blast were released in Life magazine with time stamp

• Using DA, G. I. Taylor determined energy of blast and government was upset because they thought there had been a leak of information
• Radius, $R$, of blast depends on time since explosion, $t$, energy of explosion, $E$, and density of medium, $\rho$, that explosion expands into

• $[R]=m$, $[t]=s$, $[E]=kg*\text{m}^2/\text{s}^2$, $\rho=kg/\text{m}^3$

• $R=t^pE^q\rho^k$

$$1 = 2q - 3k \quad \text{m}$$

$$0 = p - 2q \quad \text{s}$$

$$0 = q + k \quad \text{kg}$$

$q=1/5$, $k=-1/5$, $p=2/5$

$$R \propto (E/\rho)^{1/5} t^{2/5} \Rightarrow E \propto \frac{R^5 \rho}{t^2}$$

unknown constant coefficient can be determined from y-intercept of regression of log-log plot of time series
Pitfalls of Dimensional Analysis

• Miss constant factors

• Miss dimensionless ratios

• But, can get far with a good bit of ignorance!!!
Summary

- Self-similarity and fractals ➔ Power Laws
- Behavior near critical point ➔ Power Laws
- But, Power Laws ➔ near critical points
- Dimensional Analysis assumes power law form and this is partially justified by necessity of matching units

The essence of mathematics is not to make simple things complicated, but to make complicated things simple.—S. Guuder
If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is. -John von Neumann
Theories are approximations that hope to impart deeper understanding

*We all know that art [theory] is not truth. Art [theory] is a lie that makes us realize truth, at least the truth that is given us to understand. The artist [theorist] must know the manner whereby to convince others of the truthfulness of his lies.*

--Pablo Picasso
A little philosophy of science

• Many general patterns are power laws

• Can often explain these without knowledge of all the details of the system

• Art of science is knowing system well enough to have intuition about which details are important
Single prediction models are not enough

• Much better to predict value of exponent and not just that it is a power law
• To really believe a theory we need multiple pieces of evidence (possibly multiple power laws) and need to be able to predict many of these
• Understanding dynamics and some further details allows one to predict deviations from power law, and that is a very strong test and leads to very precise results
What kinds of self similarity, fractals, or critical points might be relevant for these biological scaling relationships and allometry?