Modeling Vascular Networks with Applications

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Spring 2015 Quarter
I. Newtonian fluid
II. Flow through a pipe with constant pressure gradient
III. Define and calculate flow rate, impedance, power expended, and power dissipated
IV. General time-dependent pressure gradient
   a. Fourier transforms
   b. Bessel functions
Newtonian fluids

1. The fluid must be incompressible.
2. The tube must be straight, rigid, cylindrical, and unbranched and have a constant radius.
3. The velocity of the thin fluid layer at the wall must be zero (i.e., no "slippage"). This assumption holds for aqueous solutions but not for some "plastic" fluids.
4. The flow must be laminar. That is, the fluid must move in concentric undisturbed laminae, without the gross exchange of fluid from one concentric shell to another.
5. The flow must be steady (i.e., not pulsatile).
6. The viscosity of the fluid must be constant. First, it must be constant throughout the cross section of the cylinder. Second, it must be constant in the "newtonian" sense; that is, the viscosity must be independent of the magnitude of the shear stress (i.e., force applied) and the shear rate (i.e., velocity gradient produced). In other words, the shear stress at each point is linearly proportional to its shear rate at that point.
Solution for rigid tube and constant pressure gradient
Constant pressure gradient

\[ \frac{dp}{dz} = -\frac{p}{l} \]
Navier-Stokes equation becomes

Ignoring effects of gravity and for laminar flow \( u = v_z \)

\[
\rho \frac{du}{dt} = - \frac{dp}{dz} + \mu \nabla^2 u
\]

At steady state, \( du/dt = 0 \), this becomes

\[
\mu \frac{1}{r} \frac{d}{dr} \left( r \frac{du}{dr} \right) = \frac{dp}{dz} = - \frac{p}{l}
\]
Boundary conditions

a. \( u(R,t) = 0 \)  
   no slip condition at vessel wall

b. \( \frac{du(r,t)}{dr} \bigg|_{r=0} = 0 \)  
   velocity is maximum at center of vessel
Steady-state solution for Newtonian fluid

\[ u(r) = \frac{p}{4 \mu l} \left( R^2 - r^2 \right) \]

predicts a velocity profile
Steady-state velocity profile
In 3D these are different cones of fluid flow with friction between cones.
Velocity profile develops to steady state
Once again we’re assuming non-turbulent flow and Re<2000
Calculate flow rate

General formula for flow rate

\[ \dot{Q} = \frac{1}{\rho} \frac{dM}{dt} = \frac{1}{\rho} \frac{d}{dt} \int d^3 x \ \rho = \int d\vec{S} \cdot \vec{v} = \overline{uA} \]

Expression for Newtonian fluid

\[ \dot{Q} = \frac{p \pi R^4}{8 \mu l} \]
Definition of resistance/impedance

General formula for resistance/impedance

\[ Z = \frac{p}{\dot{Q}} \]

Expression for Newtonian fluid

\[ Z = \frac{8\mu l}{\pi R^4} \quad \text{Poiseuille flow} \]
Power expended

General formula for resistance/impedance

\[ \frac{dW}{dt} = \int d^3 x (\vec{v} \cdot \nabla p) \]

Expression for Newtonian fluid

\[ \frac{dW}{dt} = \frac{p^2 \pi R^4}{8 \mu l} \]
Power dissipated

Take the dot product of the velocity, \( \vec{v} \), and integrate over the volume for both sides of the Navier-Stokes equation

\[
\frac{dW}{dt} = \int d^3x (\vec{v} \cdot \nabla p) = \int d^3x (\vec{v} \cdot \mu \nabla^2 \vec{v}) = \frac{dE}{dt}
\]

Consequently, at steady state the power expended must exactly balance the power dissipated, and for a Newtonian fluid this is

\[
\frac{dE}{dt} = -\frac{p^2 \pi R^4}{8 \mu l}
\]
Outline

1. Derive solutions for time-dependent pressure gradient like beating heart

2. Learn to use formula for general time-dependent pressure gradient by calculating
   a. constant pressure gradient as a check
   b. oscillatory pressure gradient