1. For a Newtonian fluid at steady state with laminar flow only in the \( z \) direction, solve for the velocity \( v_z \) in the following cases by including the effect of gravity, \( \rho \nabla \phi = \rho g \), where \( g \) is the acceleration due to gravity.

a. A pipe with long axis vertical to the ground and fluid flowing down towards the earth.

b. A pipe with long axis vertical to the ground and fluid flowing up away from the earth. (Hint: You should be able to do parts a and b by adapting an equation derived in class, and not solving from scratch.)

2. Discuss how solutions for velocities will be affected for a fluid flowing through a pipe with long axis horizontal to the ground for a vessel with elastic walls. You do not have to give the solution in this case, but write down the Navier-Stokes equations and explain how this differs from problem 1.

3. Thus far, we have ignored any dependence on the angle \( \theta \) by assuming rotational symmetry. This problem is to give some intuition for when this is not the case.

a. Just looking at the case of laminar flow, if we assume that \( v_z \) depends on \( \theta \) and can be expressed as \( u(r, \theta, t) \), are we able to use separation of variables in the Navier-Stokes equations. Why or why not?

b. Notice that as long as the equations \( \frac{\partial^2 v_z}{\partial \theta^2} = 0 = \frac{\partial^2 v_r}{\partial \theta^2} \) hold then our Navier-Stokes equations are unchanged. Therefore, if \( \Theta(\theta) \) is an arbitrary function of \( \theta \), solutions to the general equation

\[
\frac{\partial^2 \Theta(\theta)}{\partial \theta^2} = 0
\]

can multiply \( v_z \) and \( v_r \) such that these new velocity components are still solutions to the Navier-Stokes equation. What is the solution of \( \Theta(\theta) \) for this general equation?

c. If we think of rotational symmetry as a boundary condition, how does this constrain \( \Theta(\theta) \)? For this case, do \( v_z \) and \( v_r \) change from our solutions in class? What happens if instead we apply the boundary conditions that \( \Theta(0) = 0 \) and that \( \Theta(\pi) \) is maximal (i.e., \( \frac{\partial \Theta}{\partial \theta} \bigg|_{\theta=\pi} = 0 \)), similar to the radial boundary conditions?

4. a. The function \( \Theta(\theta) = \pi - |\pi - \theta| \) satisfies the latter boundary conditions in part c: \( \Theta(0) = 0 \) and \( \Theta(\pi) \) is maximal (i.e., \( \frac{\partial \Theta}{\partial \theta} \bigg|_{\theta=\pi} = 0 \)). Give a physical interpretation of what this function means for fluid flow and discuss what it means for turbulence? Is
this function a valid solution for $\Theta(\theta)$ such that it satisfies the general equation in problem 3b? If this function multiples $v_z$ and $v_r$ are those still valid solutions to the Navier-Stokes equations? Why did we not find this solution in problem 3c?

b. If instead we have $\frac{\partial^2 v_z}{\partial \theta^2} = -c^2 = \frac{\partial^2 v_r}{\partial \theta^2}$ where $c$ is a constant, and the equation

$$\frac{1}{\Theta(\theta)} \frac{\partial^2 \Theta(\theta)}{\partial \theta^2} = -c^2$$

how would this affect our solutions for $v_z$ and $v_r$? More specifically, how would this affect the Bessel equations we obtain for the radial component? Now, what is the solution of $\Theta(\theta)$ for this general equation? What happens if we impose the boundary conditions: $\Theta(0) = 0$ and $\Theta(\pi)$ is maximal (i.e., $\frac{\partial \Theta}{\partial \theta} |_{\theta=\pi} = 0$)?

5. a. Using the formula for derivatives of Bessel functions given in problem 6 below, and the fact that $J_{-1}(x) = -J_1(x)$, show that

$$\frac{dJ_0(x)}{dx} = -J_1(x)$$

b. Using the recursion relation and the formula for derivatives in part problem 7, show that

$$\frac{d(xJ_1(x))}{dx} = xJ_0(x)$$

so that

$$\int dxxJ_0(x) = xJ_1(x)$$

EXTRA CREDIT

6. To learn about Bessel functions, start with the recursion relation

$$J_n(x) = \frac{x}{2n} (J_{n-1}(x) + J_{n+1}(x))$$

and substitute this into the differential equation for $J_n(x)$ and show that by using the Bessel equations for $J_{n-1}(x)$ and $J_{n+1}(x)$ and repeated use of the recursion relation, you can derive the formula for derivatives

$$2 \frac{dJ_n(x)}{dx} = J_{n-1}(x) - J_{n+1}(x)$$