From this point the logic of random walks, we have the rule

\[ P(i, n) = \frac{1}{2} P(i-1, n-1) + \frac{1}{2} P(i+1, n-1) \]

So, we literally just add the probabilities from sites to the left and right in the previous time step and divide by 2. At each site, we have a choice of left or right (like a coin flip of heads or tails) to go to the next step with probability \( \frac{1}{2} \).
2. (a) From the previous problem or our rule, it is clear that we divide by 2 at each case time step to get \((\frac{1}{2})^n\). Moreover, we sum across all possible paths, equivalent to taking all possible combinations of left and right (binary choice) steps. Thus, this is a binomial, and by inspection and symmetry \([P(i,n) = P(n-i,n)]\) and being at odd sites \(i\) at odd time steps \(n\), and even time steps \(i\) at even time steps \(n\), we can choose \(k = \frac{n+i}{2}\) or \(k = \frac{n-i}{2}\).

Either way, we have

\[
P(i,n) = \frac{1}{2^n} \binom{n}{\frac{n+i}{2}} = \frac{1}{2^n} \frac{n!}{\left(\frac{n+i}{2}\right)! \left(\frac{n-i}{2}\right)!} = \frac{1}{2^n} \frac{n!}{(n+i)! (n-i)!}
\]

\[
\sum_{i=0}^{n} P(i,n) = \sum_{i=0}^{n} \frac{1}{2^n} \binom{n}{\frac{n+i}{2}} = \sum_{i=0}^{n} \binom{n}{\frac{n+i}{2}} \left(\frac{1}{2}\right)^{\frac{n+i}{2}} \left(\frac{1}{2}\right)^{\frac{n-i}{2}} = \sum_{i=0}^{n} \binom{n}{\frac{n+i}{2}} \frac{1}{2}^i
\]

\[
= (\frac{1}{2} + \frac{1}{2})^n = 1^n = 1.
\]

\(\checkmark\)
(3) \( P(i, n) = \frac{1}{2} \left[ P(i-1, n-1) + P(i+1, n-1) \right] \)

\[
= \frac{1}{2} \left[ \frac{1}{2^{n-1}} \frac{(n-1)!}{(n-i-1)! \left( \frac{n-i+1}{2} \right)!} + \frac{1}{2^{n-1}} \frac{(n-1)!}{(n-i+1)! \left( \frac{n-i+1}{2} \right)!} \right]
\]

\[
= \frac{1}{2^n} (n-1)! \left[ \frac{1}{(n-i)! \left( \frac{n+i}{2} \right)!} + \frac{1}{(n-i)! \left( \frac{n-i}{2} \right)!} \right]
\]

\[
= \frac{1}{2^n} \frac{(n-1)!}{(n-i)! \left( \frac{n+i}{2} \right)!} \left[ \frac{n-i}{2} + \frac{i}{2} \right] = n
\]

\[
= \frac{1}{2^n} \frac{n \cdot (n-1)!}{(n-i)! \left( \frac{n+i}{2} \right)!} = \frac{1}{2^n} \frac{n!}{(n+i)! \left( \frac{n-i}{2} \right)!}
\]
\( \text{4) From our formula } P(i/n) = P(-i/n), \text{ so} \\
\text{c \cdot P(i/n) - c \cdot P(-i/n) = 0.} \\
\text{Thus, } \sum_{i = 0}^{n} c \cdot P(i/n) = 0 \cdot P(0/n) = 0. = \langle i \rangle \\
\text{all } i \neq 0 \text{ terms} \\
\text{(can be matched up for } i \text{ and } -i) \\
\text{to cancel.} \\
\text{5) Var}(i) = \langle i^2 \rangle - \langle i \rangle^2 = \langle i^2 \rangle \\
\text{because } \langle \psi \rangle = 0 \\
\text{From homework 1, the variance of a binomial is} \\
np(1-p). \text{ Our probability distribution has been} \\
\text{shown to be a binomial with } p = \frac{1}{2}, \text{ so the variance} \\
is n \frac{1}{2} (1-\frac{1}{2}) = \frac{n}{4}. \text{ However, since this is a variance} \\
\text{and we take } i \text{ Var, } -n \rightarrow n, \text{ then there are really } 2i \text{ states} \\
in our previous notation, so this is really } \frac{\text{Var}(002)}{\text{Var}(\frac{1}{2})} \\
\text{states} \\
\langle (\frac{1}{2})^2 \rangle = \langle \frac{i^2}{4} \rangle \\
\Rightarrow \frac{\langle i^2 \rangle}{4} = \frac{n}{4} \Rightarrow \langle i^2 \rangle = n \Rightarrow \langle i \rangle = 0 \}
What is the expected time for 3 loci to coalesce in a population of 2N loci? Assume only genetic drift and thus the Wright-Fisher model. Also assume \(2N \gg t \gg 1\).

**Solution:**

\[
\text{Chance that any pair does not coalesce at previous time step} = 1 - \frac{1}{2N} \quad \text{(as derived in class)}
\]

There are \(\binom{3}{2} = \frac{3!}{2!1!} = 3\) pairs, so the chance that no pair coalesced at previous time step is:

\[
(1 - \frac{1}{2N})^3
\]

By definition, therefore, the chance a pair did coalesce at previous time step is:

\[
1 - (1 - \frac{1}{2N})^3
\]

The chance of coalescing till time \(t\) steps in the past is thus:

\[
\left(1 - \left(1 - \frac{1}{2N}\right)^3\right)^t \left[1 - \left(1 - \frac{1}{2N}\right)^3\right]
\]

\[
\sim \frac{3^t}{2N^t} \left(1 - \frac{3}{2N}\right) \sim \frac{3}{2N} e^{-\frac{3t}{2N}}
\]

Finally, note that chance that all three coalesced at previous time step is found by realizing one locus has a parent, and other two loci each have a fixed chance of having that same (fixed) locus as a parent.

Thus, \(\left(\frac{1}{2N}\right)^2 \ll 1 - \left(1 - \frac{1}{2N}\right)^3 \sim \frac{3}{2N}\) because \(\frac{1}{N^2} \ll \frac{1}{N}\)

\(0\) for \(N\) large

Note normalization is correct such that this integrates to 
\[
\int_0^\infty \frac{3}{2N} e^{-\frac{3t}{2N}} dt = 1
\]

so this is a p.d.f.
Consequently, for $N > 1$, we ignore the probability that 3 loci coalesce in one time step.

Hence, the expected time until one pair coalesces is

\[ \langle t_1 \rangle = \frac{3}{2N} \int_0^\infty t e^{-\frac{3}{2N}t} \, dt \]

\[ = \int_0^\infty t (-\frac{3}{2N} e^{-\frac{3}{2N}t}) \, dt \]

\[ = \left[ -t e^{-\frac{3}{2N}t} \right]_0^\infty + \int_0^\infty e^{-\frac{3}{2N}t} \, dt \]

\[ = -\frac{2N}{3} \left[ e^{-\frac{3}{2N}t} \right]_0^\infty + \frac{2N}{3} \]

Our picture is now

\[ \begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
9 \\
10
\end{array} \]

We have calculated the expected time to reach $t_1$. Now we need to know the expected time to reach $t_2$.

At $t_2$, we are now down to 2 individuals. We already calculated in class the expected time for 2 individuals to coalesce, so

\[ \langle t_2 - t_1 \rangle = 2N \]
Combining all this, the expected time for 3 loci to coalesce is
\[
\sqrt{T_3} = \langle t_1 \rangle + \langle t_2 - t_1 \rangle = \frac{2N}{3} + 2N = 2N(\frac{1}{3} + 1) = \frac{8N}{3} = 2\frac{2}{3}N
\]

Note that this is > 2N but much less than 2·2N = 4N which makes sense.

5. What is the expected time in the past for k loci to coalesce in a population of 2N loci? Assume only genetic drift and the Wright-Fisher model. Also assume 2N >> t >> 1.

Solution:

Using same logic as above, we only consider changes of one pair at a time coalescing because other terms are \(O(1)\) with \(k^2N\) which is small compared with \(O(1)\), which is chance of 1 pair coalescing.

Chance none of \(k\) pairs coalesce is \((1 - \frac{1}{2N})^{k\choose 2}\)

Chance 1 pair coalesce is \(1 - (1 - \frac{1}{2N})^{k\choose 2}\)

Chance 1 pair coalesce at \(t+1 = \left[1 - (1 - \frac{1}{2N})^{k\choose 2}\right]^{t+1} \times \left[1 - (1 - \frac{1}{2N})^{k\choose 2}\right]^{t} \approx e^{-\frac{k^2}{2N} \left(1 - \frac{1}{2N} + \left(\frac{1}{2N}\right)^2\right)} = \frac{\left(\frac{1}{2N}\right)^{\frac{k}{2}t}}{2N} e^{-\frac{k}{2N} t}

Note that \(\int_0^\infty \frac{k}{2N} e^{-\frac{k}{2N} t} dt = 1\) so this is a PDF.
When the expected time to coalescence of the first pair is
\[ \langle t_1 \rangle = \frac{1}{2N} \int_0^\infty t e^{-\frac{3t}{2N}} dt = \int_0^\infty t (-\frac{1}{3} t e^{-\frac{3t}{2N}}) dt \]
\[ = \left[ t e^{-\frac{3t}{2N}} \right]_0^\infty + \int_0^\infty e^{-\frac{3t}{2N}} dt \]
\[ = -\frac{2N}{3} \left[ e^{-\frac{3t}{2N}} \right]_0^\infty = -\frac{2N}{3} \left[ \frac{2N}{3} \right] \]

By definition, our next time interval is the expected time for
\[ \langle t_2 - t_1 \rangle = \frac{2N}{(k-1)} \]

Seeing the pattern
\[ \langle t_k \rangle = \langle t_1 \rangle + \langle t_2 - t_1 \rangle + \langle t_3 - t_2 \rangle + \ldots + \langle t_k - t_{k-1} \rangle \]
\[ = \sum_{i=1}^{k} \frac{2N}{(i-1)!} = 2N \sum_{i=2}^{k} \frac{1}{i(i-1)!} \]

When \( k \geq 2 \), largest terms are \( \frac{2N}{(i-1)!} \leq \frac{2N}{2!} \leq \frac{2N}{1 \cdot 2} < 2! = 2 \),
so when \( k \geq 2 \),
\[ \langle t_k \rangle \leq 2N \cdot 2 
\]
Or \( \langle T_k \rangle = 4N \sum_{i=2}^{k} \frac{1}{i(i-1)} \)

\[
\langle T_k \rangle = 4N \sum_{i=2}^{k} \left( \frac{1}{i-1} - \frac{1}{i} \right) = 4N \left[ (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + (\frac{1}{3}-\frac{1}{4}) + \ldots + (\frac{1}{k-1}-\frac{1}{k}) \right]
\]

\[= 4N \left[ 1 - \frac{1}{k} \right] = \frac{4(k-1)N}{k} \]

Telescoping series, so cancel terms

Check: \( k = 2 \Rightarrow \langle T_2 \rangle = \frac{2\left(2^2-1\right)}{2} = 2N \) \( \checkmark \)

\( k = 3 \Rightarrow \langle T_3 \rangle = \frac{4\left(3^2-1\right)}{3} = \frac{8N}{3} \) \( \checkmark \)

Large \( k \) limit (\( k \geq 1 \)): As \( k \to \infty \), \( \frac{k-1}{k} = \frac{k(1-\frac{1}{k})}{k} \) because \( \frac{1}{k} \) is small.

\[\Rightarrow \langle T_k \rangle = 4N = 2 \langle T_2 \rangle \]
when \( k \to \infty \).

Total time to coalesce infinitely more loci is the same as the time to coalesce just the first pair!
Ex: a Credit (0)

If \( u = 1 + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \)
\( v = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \)
\( w = \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \)

Prove that \( u^3 + v^3 + w^3 - 3uvw = 1 \)

We differentiate the left side to find

\[ (1) \quad 3u^2u' + 3v^2v' + 3w^2w' - 3uvw' - 3uvw - 3uvw'w = 0 \]

From the above, it immediately follows that \( v' = u, w' = v, u' = w \), so we have

\[ 3u^2w + 3v^2u + 3w^2v - 3uvw - 3wv^2 - 3u^2w = 0 \]

Integrating \((1) = 0\) gives

\[ u^3 + v^3 + w^3 - 3uvw + \text{Constant} = 0 \]

At \( x = 0 \), \( u = 1, v = 0, w = 0 \), so \( u^3 + v^3 + w^3 - 3uvw = 1 \)

Since the right side is a constant, this must be true everywhere, so

\[ u^3 + v^3 + w^3 - 3uvw = 1 \]
Substitute \( P_n = P_0 \sum_{i=1}^{n-1} \frac{b_i}{d_i + 1} \) into \( P_{n+1} \) to show

\[
P_{n+1} \Delta n_{n+1} + P_{n-1} b_{n-1} - P_n (b_n + \Delta n) = 0.
\]

\[
P_0 \left( \sum_{i=1}^{n} \frac{b_i}{d_i + 1} \right) \Delta n_{n+1} + P_0 \left( \sum_{i=1}^{n-1} \frac{b_i}{d_i + 1} \right) b_{n-1} - P_0 \left( \sum_{i=1}^{n-1} \frac{b_i}{d_i + 1} \right) b_n = 0.
\]

\[
= P_0 \left( \sum_{i=1}^{n-2} \frac{b_i}{d_i + 1} \right) b_{n-1} b_n + P_0 \left( \sum_{i=1}^{n-2} \frac{b_i}{d_i + 1} \right) b_{n-1} - P_0 \left( \sum_{i=1}^{n-2} \frac{b_i}{d_i + 1} \right) b_n - P_0 \left( \sum_{i=1}^{n-2} \frac{b_i}{d_i + 1} \right) b_n = 0.
\]
\[ \frac{dF}{dt} = D \frac{d^2 F}{dx^2} \]

\[ N = \frac{x}{\sqrt{t}} \] is our fundamental parameter of space versus time.

Guess a solution \( F(x,t) = t^p g(n) \).

Substitute into equation,

\[ \frac{dF}{dt} = \frac{d}{dt} \left( t^p g(n) \right) = p t^{p-1} g(n) + t^p \frac{dg}{dn} \]

by chain rule

\[ \frac{d}{dt} \left( \frac{x}{\sqrt{t}} \right) = \frac{x}{2t^{3/2}} - \frac{1}{2} \frac{x}{t^{3/2}} = -\frac{1}{2t} n \]

\[ = p t^{p-1} g(n) + t^p \frac{dg}{dn} \]

\[ = t^{p-1} \left[ \frac{n g(n)}{2} \right] \frac{dg}{dn} \]

\[ = t^{p-1} \left[ n \frac{dg}{dn} \right] \]
\[ \frac{d^2 f}{dx^2} = \frac{d^2 (t^p g(x))}{dx^2} = t^p \frac{d}{dx} \left( \frac{dg(x)}{dx} \right) \]

by chain rule
\[ \frac{dg(x)}{dx} = \frac{1}{t^p (x)} \]
\[ \frac{d}{dx} \left( \frac{1}{t^p (x)} \right) = \frac{1}{t^p (x)^2} \]

because it does not depend on x we can move it through other derivative of x

Putting this all together, we have from (5)

\[ t^{p-1} \left[ p g(x) + \frac{2}{t} \frac{dg(x)}{dx} - D \frac{d^2 g}{dx^2} \right] = 0 \]

\[ = \left[ p g(x) - \frac{1}{t} \frac{dg(x)}{dx} - D \frac{d^2 g}{dx^2} \right] = 0 \]

Ordinary Differential Equation in x
Not Partial. Called Similarity Solution

(b) If \( p = -\frac{1}{2} \), we can write this
as a single derivative

\[ -\frac{1}{2} g(x) \frac{\partial}{\partial x} \left( \frac{dg(x)}{dx} \right) - D \frac{d^2 g}{dx^2} = 0 \]
\[ -\frac{1}{2} n g(n) - D \frac{dg}{da} = A e^{\text{constant}}. \]

Assuming \( g^{(0)} = f'(0) = 0 \) we have \( A = 0 \), so \( g(n) = 0 \) but \( n = 0 \) and \( \frac{dg}{da} = 0 \).

\[ \frac{1}{2} n g(n) + D \frac{dg}{da} = 0 \]

\[ \Rightarrow 2D \int \frac{dg(n)}{g(n)} = -\int d(n^2) \]

\[ 2D \ln g(n) = -\frac{n^2}{2} + C \]

\[ \Rightarrow \ln g(n) = -\frac{n^2}{4D} + C' e^{C} \]

\[ \Rightarrow g(n) = A e^{-\frac{n^2}{4D}} = A e^{-\frac{x^2}{4Dt}} \]

\[ \Rightarrow f(x,t) = B A e^{-\frac{x^2}{4Dt}} \]

\[ \square \]

This cannot be found from separation of variables (S of V).

So, V assumes \( f(x,t) = h_1(x) h_2(t) \) and the solution above cannot be written in this form because it is not separable.
Given an infinite amount of time, every node will be visited, in which case we are done, unless the heart is part of a loop, in which case we are done, or the path goes into a loop somewhere else, and the heart is not part of this new loop. Thus, we must prove that the latter (Case 3) cannot happen.

To imagine the two smallest instances of this network, we can draw them graphically as

![Diagram](image)

As suggested, we will look at the complementary or "dual" network constructed by placing new nodes at the midpoints of the vessels, and then connecting nodes (branch points) that have vessels join at the same node.
Hence, a standard unit of our network for any size is

Which maps to the dual network

Or, for the smallest, full graph

Moreover, there are two paths through any standard unit

(\(L \rightarrow R\)) or (\(R \rightarrow L\))

Only straight lines are now possible. So, if the node is part of a loop there are two possibilities
For the path to go into a loop that it is not part of, it would require a vertex like X, so that O. However, we have shown that this is not possible in a diagram, and consequently, that it is not possible to enter a loop of which you are not already a part. Therefore, we have shown that case 3 is impossible, so it must be that blood returns to the heart.